Complexity Qual Spring 2021
(provide proofs for all answers)
(90 Points Total)

In the following:

- A TM is a Turing machine.
- P is the class of languages accepted by deterministic TMs running in time $n^{O(1)}$.
- NP is the class of languages accepted by nondeterministic TMs running in time $n^{O(1)}$.
- **Provide a detailed proof for each of your answers.**

Problem 1: Regular and Context-free Languages (15 Points Total):
(a) Consider the language $C = \{w | w = a^{k!} | k = 1, 2, \ldots\}$. Prove or disprove that C regular.

(b) Give a context-free grammar that recognizes the set of words over the alphabet $\{0{,}1\}$ such that the number of 0s in even-numbered positions equals the number of 1s in odd-numbered positions.

Problem 2: Decidability (15 Points Total): Let MOVES-LEFT be the set of strings $\langle M, w \rangle$ such that M is a TM whose head moves left at some point in its computation on input w. Is MOVES-LEFT is decidable? Prove your answer.

Problem 3: Recursive and Recursively Enumerable Languages (15 Points Total):

(a) Show that if L is recursively enumerable, then there is a Turing machine (TM) that enumerates L without repeating an element of L.

(b) Show that L is recursive if and only if there is a TM that enumerates the strings of L in *length increasing* fashion.

Problem 4: Closure Properties of P and NP (15 Points Total):
Define the *Kleene star* of a language L to be $L^* = \{x_1 \cdot x_2 \dots x_k | k \geq 0; x_1, \dots, x_k \in L\}$.
(a) Show that NP is closed under Kleene star, i.e., if $L \in$ NP then $L^* \in$ NP.

(b) Is P also closed under Kleene star? Give a proof or disproof. Hint: Use dynamic programming.

Problem 5: NP Completeness (15 Points Total): City X is implementing a social distancing policy during this pandemic. Specifically, in a park which is described by an undirected graph G (with n vertices and m edges), the minimum distance between two people needs to be at least 2 at any time. There are k people in the park, person i wants to go from vertex s_i to vertex t_i (these vertices are not adjacent to each other). At each step, a person can move along one edge in the graph, or stand still. The *SOCIAL DISTANCING* problem is to decide whether there is a way for everyone to go from their respective starting point s_i to their ending point t_i within p steps, while still observing the social distancing policy. *INDEPENDENT SET* is a classical NP-complete problem, where there is an undirected graph G with n vertices and m edges, the goal is to decide whether there exists k vertices such that no two vertices are connected by an edge. Based on the fact that INDEPENDENT SET is NP-complete, prove SOCIAL DISTANCING is NP-hard.

Problem 6: Randomized Computation (15 Points Total): Let L be a decision problem that has a polynomial time randomized algorithm A. For any input x, the probability that A is correct on x is at least 2/3. Suppose we want to solve all inputs of size n, show that there is a circuit of size poly(n) that can be correct for all inputs of size n. (**Hint:** You can repeat A to improve its probability of being correct.)